BI-TP 94/61

ALGORITHMIC CALCULATION OF TWO-LOOP FEYNMAN DIAGRAMS *

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Abstract

In a recent paper [1] a new powerful method to calculate Feynman diagrams was proposed. It consists in setting up a Taylor series expansion in the external momenta squared. The Taylor coefficients are obtained from the original diagram by differentiation and putting the external momenta equal to zero. It was demonstrated that by a certain conformal mapping and subsequent resummation by means of Padé approximants it is possible to obtain high precision numerical values of the Feynman integrals in the whole cut plane. The real problem in this approach is the calculation of the Taylor coefficients for the arbitrary mass case. Since their analytic evaluation by means of CA packages uses enormous CPU and yields very lengthy expressions, we develop an algorithm with the aim to set up a FORTRAN package for their numerical evaluation. This development is guided by the possibilities offered by the formulae manipulating language FORM [2].

* to be published in the proceedings of the workshop on Computer Algebra in Science and Engineering, held at Zentrum für Interdisziplinäre Forschung (ZiF), Bielefeld, Germany, August 1994; World Scientific, J.Fleischer, J.Grabmeier, F.W. Hehl and W.Küchlin, editors.

† Supported by Bundesministerium für Forschung und Technologie.

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1. Introduction

Standard-model radiative corrections of high accuracy have obtained growing attention lately in order to cope with the increasing precision of LEP experiments [3]. In particular two-loop calculations with nonzero masses became relevant [4]. While in the one-loop approach there exists a systematic way of performing these calculations [5], in the two-loop case there does not exist such a developed technology and only a series of partial results were obtained [6], [7] but no systematic approach was formulated.

Our approach consists essentially in performing a Taylor series expansion in terms of external momenta squared and analytic continuation into the whole region of kinematical interest. Simple as this may sound, there are some unexpected methodical advantages compared to other procedures.

Considering a Taylor series expansion in terms of one external momentum squared, q^2 say, the differential operator by the repeated application of which the Taylor coefficients are obtained, subsequently setting $q = 0$, is

$$\Box_q = \frac{\partial^2}{\partial q_\mu \partial q^\mu}. \tag{1}$$

Such expansions were considered in [8], Padé approximants were introduced in [9] and in Ref. [1] it was demonstrated that this approach can be used to calculate Feynman diagrams on their cut which, concerning physics, is the most interesting case. The above Taylor coefficients are essentially "bubble diagrams", i.e. diagrams with external momenta equal zero. They are essentially the same (after partial fraction decomposition) for two-point, three-point, ... functions for a given number of loops and we stress that it is indeed a great technical simplification to have to perform integrals only for external momenta equal zero even if these integrals contain now arbitrary high powers of the scalar propagators. For their calculation recurrence relations are quite effective ([8],[10], see Sect.6). On this basis we develop our algorithm.

2. Expansion of three-point functions in terms of external momenta squared

Here we have two independent external momenta in $d = 4 - 2\varepsilon$ dimensions. The general expansion of (any loop) scalar 3-point function with its momentum space representation $C(p_1, p_2)$ can be written as

$$C(p_1,p_2) = \sum_{l,m,n=0}^{\infty} a_{lmn}(p_1^2)^l(p_2^2)^m(p_1p_2)^n = \sum_{L=0}^{\infty}\sum_{l+m+n=L} a_{lmn}(p_1^2)^l(p_2^2)^m(p_1p_2)^n, \tag{2}$$

where the coefficients a_{lmn} are to be determined from the given diagram. They are obtained by applying the differential operators $\Box_{ij} = \frac{\partial}{\partial p_{i\mu}}\frac{\partial}{\partial p_j^\mu}$ several times to both sides of (2).

This procedure results in a system of linear equations for the a_{lmn}. For fixed L (see equation (2)) we obtain a system of $(L+1)(L+2)/2$ equations of which, however, maximally $[L/2]+1$ couple ($[x]$ standing here for the largest integer $\leq x$). These linear equations are easily solved with REDUCE [11], e.g., for arbitrary d.

For the purpose of demonstrating the method, we confine ourselves to the case $p_1^2 = p_2^2 = 0$, which is e.g. physically realized in the case of the Higgs decay into two photons ($H \to \gamma\gamma$) with p_1 and p_2 the momenta of the photons. In this case only the coefficients a_{00n} are needed. They are each obtained from a "maximally coupled" system of $[n/2]+1$ linear equations. Solving these systems of equations we obtain a sequence of differential operators (Df's) which project out from the r.h.s. of (2) the coefficients a_{00n}:

$$Df_{00n} = \sum_{i=1}^{[n/2]+1} \frac{(-4)^{1-i}\Gamma(d/2+n-i)\Gamma(d-1)}{2\Gamma(i)\Gamma(n-2i+3)\Gamma(n+d-2)\Gamma(n+d/2)}(\Box_{12})^{n-2i+2}(\Box_{11}\Box_{22})^{i-1}, \tag{3}$$

where the sum of the exponents of the various $\Box'$ s is equal n. Applying the operator Df_{00n} to the (scalar) momentum space integral $C(p_1, p_2)$ and putting the external momenta equal to zero, yields the expansion coefficients a_{00n}.

In the two-loop case we consider the scalar integral ($k_3 = k_1 - k_2$, see also Fig. 1)

$$C(m_1, \cdots, m_6; p_1, p_2) \\ = \frac{1}{(i\pi^2)^2} \int \frac{d^4k_1 d^4k_2}{((k_1+p_1)^2-m_1^2)((k_1+p_2)^2-m_2^2)((k_2+p_1)^2-m_3^2)((k_3+p_2)^2-m_4^2)(k_2^2-m_5^2)(k_3^2-m_6^2)}, \tag{4}$$

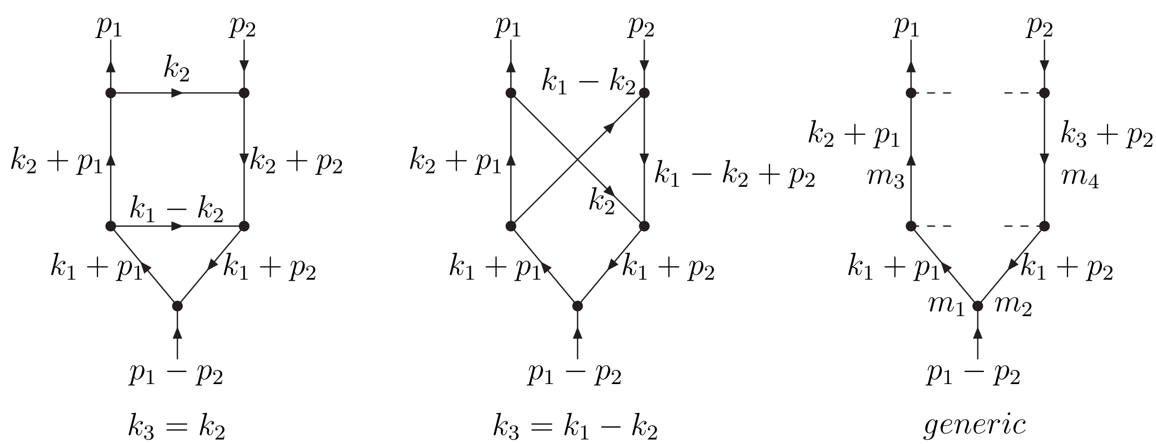

Figure 1: Planar and non-planar scalar vertex diagrams and their kinematics

Introducing the abbreviations $c_1 = k_1^2 - m_1^2, c_2 = k_1^2 - m_2^2, c_3 = k_2^2 - m_3^2, c_4 = k_3^2 - m_4^2$ and $c_5 = k_2^2 - m_5^2, c_6 = k_3^2 - m_6^2$, we have ($c_5$ and c_6 do not enter the differentiation since for the planar as well as for the non-planar diagram they occur in (4) as such)

$$(i\pi^2)^2 a_{00n} = \frac{2^n}{n+1} \int d^4k_1 d^4k_2 F_n \cdot \frac{1}{c_1\ c_2\ c_3\ c_4\ c_5\ c_6}, \tag{5}$$

where for the *planar* diagram ($k_3 = k_2$ in c_4 and $k_3 = k_1 - k_2$ in c_6)

$$F_n = \sum_{\nu=0}^{n} c_1^{-(n-\nu)} c_3^{-\nu} \sum_{\nu'=0}^{n} c_2^{-(n-\nu')} c_4^{-\nu'} \cdot A^n_{\nu\nu'}(k_1, k_2), \tag{6}$$

and

$$A^n_{\nu\nu'}(k_1, k_2) = \sum_{0 \le 2\mu \le \nu+\nu' \le n+\mu} a^{n\mu}_{\nu\nu'} (k_1^2)^{n-(\nu+\nu')+\mu} (k_2^2)^{\mu} (k_1\ k_2)^{\nu+\nu'-2\mu}, \tag{7}$$

$a^{n\mu}_{\nu\nu'}$ being rational numbers with the properties

$$a^{n\mu}_{\nu\nu'} = a^{n\mu}_{\nu'\nu} \qquad \text{and} \qquad \sum_{\mu} a^{n\mu}_{\nu\nu'} = 1, \tag{8}$$

The above is now essentially the basis for the algorithm we are going to develop: to calculate the Taylor coefficients of integrals like (4) according to (5) - (8) and to reduce them to integrals of the type

$$V_B(\alpha,\beta,\gamma,m_1,m_2,m_3) = (-1)^{(\alpha+\beta+\gamma)} \int \frac{d^dk_1 d^dk_2}{(k_1^2-m_1^2)^\alpha (k_2^2-m_2^2)^\beta ((k_1-k_2)^2-m_3^2)^\gamma}, \quad (9)$$

which in turn will be reduced by means of recurrence relations to $V_B(1,1,1,m_1,m_2,m_3)$. Accordingly our algorithm is performed in the following three steps:

- First of all the coefficients $a^{n\mu}_{\nu\nu'}$ in (7) and the corresponding ones for the non-planar diagram can be evaluated in terms of multiple sums over Γ-functions. While (7) has been obtained in [1] by inspection of FORM output and some lower coefficients could be read off explicitly, in Sect. 4 we give a proof of this representation by construction, which also yields the $a^{n\mu}_{\nu\nu'}$.

- From (9) it becomes clear that the numerator scalar products in (7) must be eliminated and/or a partial fraction decomposition of products of scalar propagators with the same integration momentum k_1, k_2 or k_3 but different masses must be performed. Substituting, e.g., $k_1^2 = c_i + m_i^2$ (i=1,2) one cancels k_1^2. Similarly one proceeds for k_2^2 and k_3^2. For k_1k_2 in (7) one writes for the *planar* diagram

$$k_1k_2 = \frac{1}{2}(k_1^2+k_2^2-m_6^2) - \frac{1}{2}c_6 \equiv \frac{1}{2}k^2 - \frac{1}{2}c_6 \quad (10)$$

  and by stepwise reducing higher powers ($\nu+\nu'-2\mu=\lambda$)

$$2^\lambda (k_1k_2)^\lambda = (k^2)^\lambda - \left[(k^2)^{\lambda-1} + 2k_1k_2(k^2)^{\lambda-2} + \ldots + (2k_1k_2)^{\lambda-1}\right] c_6. \quad (11)$$

  In the second term c_6 cancels after insertion into (5) so that only factorized one-loop contributions are obtained from it (i.e. the integral (5) factorizes into integrals over k_1 and k_2 separately). Moreover, in the square bracket of (11) only even powers of k_1k_2 contribute after integration. The "genuine" two-loop contributions are then obtained by replacing k_1k_2 in (7) by $\frac{1}{2}(k_1^2+k_2^2-m_6^2)$ according to the first term in (11). The problem of cancelling k_1k_2 is more complicated for the non-planar diagram, as will be demonstrated in Sect. 5.

- The evaluation of the integrals (9) is supposed to be performed in terms of recurrence relations, thus reducing them to known two-loop "master" integrals, as will be described in Sect. 6. If one of the indices α,β,γ in (9) is ≤ 0 (i.e. in our notation the corresponding scalar propagator occurs with positive power in the numerator), the integral can be expressed again in terms of factorized one-loop integrals and simple explicit representations can be found in this case. An example is also given at the end of Sect. 6.

3. The method of analytic continuation

Before entering the details of the calculation, we want to give a motivation for the above algorithm, i.e. the main interest in Feynman diagrams is for the values on their cut and we have to demonstrate how to obtain these from the Taylor expansion.

Assume, the following Taylor expansion of a scalar diagram or a particular amplitude is given:

$$C(p_1,p_2,\ldots) = \sum_{m=0}^{\infty} a_m y^m \equiv f(y) \quad (12)$$

and the function on the r.h.s. has a cut for $y \geq y_0$. In the above case of $H \to \gamma\gamma$ one introduces $y = \frac{q^2}{4m_t^2}$ with $q^2 = (p_1 - p_2)^2$ as adequate variable with $y_0 = 1$.

Our proposal for the evaluation of the original series is in a first step a conformal mapping of the cut plane into the unit circle and secondly the reexpansion of the function under consideration into a power series w.r.to the new conformal variable. A variable often used [12] is

$$\omega = \frac{1 - \sqrt{1 - \frac{y}{y_0}}}{1 + \sqrt{1 - \frac{y}{y_0}}}. \tag{13}$$

Considering it as conformal transformation, the y-plane, cut from y_0 to $+\infty$, is mapped into the unit circle and the cut itself is mapped on its boundary, the upper semi-circle corresponding to the upper side of the cut. The origin goes into the point $\omega = 0$.

After conformal transformation it is suggestive to improve the convergence of the new series w.r.to ω by applying one of the numerous summation methods [13],[14] most suitable for our problem. We obtained the best results with the Padé method and partially also with the Levin v transformation. The expansion of $f(y)$ in terms of ω is:

$$f(y(\omega)) = \sum_{s=0}^{\infty} \omega^s \phi_s, \tag{14}$$

where

$$\begin{aligned} \phi_0 &= a_0 \\ \phi_s &= \sum_{n=1}^{s} a_n (4y_0)^n \frac{\Gamma(s+n)(-1)^{s-n}}{\Gamma(2n)\Gamma(s-n+1)}, \qquad s \geq 1. \end{aligned} \tag{15}$$

Eq.(14) will be used for the analytic continuation of f into the region of analyticity ($y < y_0$; observe that the series (12) converges for $|y| < y_0$ only) and in particular for the continuation on the cut ($y > y_0$). In this latter case we write

$$\omega = \exp[i\xi(y)], \qquad \text{with} \quad \cos\xi = -1 + 2\,\frac{y_0}{y} \tag{16}$$

and hence

$$f(y) = a_0 + \sum_{n=1}^{\infty} \phi_n \exp in\xi(y) \tag{17}$$

In any case we have $|\omega| \leq 1$ and we will show in the following how to sum the above series.

Padé approximations are indeed particularly well suited for the summation of the series under consideration. In the case of two-point functions they could be shown in several cases (see e.g. [9]) to be of Stieltjes type (i.e. the spectral density is positive). Under this condition the Padé's of the original series (12) are guaranteed to converge in the region of analyticity. For the three-point function under consideration ($H \to \gamma\gamma$), however, the obtained result shows that the series is not of Stieltjes type (i.e. the imaginary part changes sign on the cut).

Having performed the above ω transformation (13), however, it is rather the Baker-Gammel-Wills *conjecture* (see [15]), which applies.

A convenient technique for the evaluation of Padé approximants is the ε-algorithm of [13]. In general, given a sequence $\{S_n | n = 0, 1, 2, \ldots\}$, one constructs a table of approximants using

$$T(m,n) = T(m-2, n+1) + 1/\left\{T(m-1, n+1) - T(m-1, n)\right\}, \tag{18}$$

with $T(0,n) \equiv S_n$ and $T(-1,n) \equiv 0$. If the sequence $\{S_n\}$ is obtained by successive truncation of a Taylor series, the approximant $T(2k,j)$ is identical to the $[k+j/k]$ Padé approximant [13], derived from the first $2k+j+1$ terms in the Taylor series.

We present results for the two-loop three-point scalar (*planar*) integral with the kinematics of the decay $H \to \gamma\gamma$. We study the integral (4) with $m_6 = 0$ and all other masses $m_i = m_t (i = 1,..,5)$. In this special case all Taylor coefficients can be expressed in terms of Γ -functions. For a list of the first coefficients a_{00n} ($\equiv a_n; n = 0, ..., 28$) see [1].

Table 1: Results on the cut ($q^2 > 4m_t^2$) in comparison with [16].

q^2/m_t^2	[10/10] Re	[10/10] Im	[14/14] Re	[14/14] Im	Ref.[16] Re	Ref.[16] Im
4.01	11.926	12.66	11.935	12.699	11.9347(1)	12.69675(8)
4.05	5.195	10.48	5.1952	10.484	5.1952(1)	10.4836(4)
4.10	2.6624	9.095	2.66245	9.0955	2.66246(2)	9.0954(2)
4.20	0.5161	7.4017	0.516039	7.401640	0.51604(5)	7.40163(4)
4.50	- 1.42315	4.77651	- 1.42315097	4.77651003	- 1.423122(9)	4.776497(9)
5.0	- 1.985805	2.758626	- 1.985804823	2.758626375	- 1.98580(2)	2.758625(2)
6.0	- 1.7740540	1.1232494	- 1.774053979	1.123249363	- 1.77405(1)	1.123250(6)
7.0	- 1.4192404	0.4807938	- 1.419240377	0.4807938045	- 1.419240(5)	0.480794(9)
8.0	- 1.13418526	0.1784679	- 1.134185262	0.1784687866	- 1.134184(1)	0.178471(2)
10.0	- 0.75694327	- 0.06154833	- 0.7569432708	- 0.0615483234	- 0.756943(1)	- 0.061547(1)
40.0	- 0.045853	- 0.0645673	- 0.045852780	- 0.0645672604	- 0.04585286(7)	- 0.0645673(9)
400.0	+ 0.000082	- 0.002167	+ 0.00008190	- 0.0021670	0.0000818974(3)	- 0.002167005(3)

Results for this kinematics on the cut are given in Table 1. The process $H \to \gamma\gamma$ was investigated before in Ref. [16]. For the master integral under consideration in [16] all integrations but one could be performed analytically and only the last one had to be done numerically (hence the high precision achieved).

Similarly, high precision is obtained on the cut in our approach, as is demonstrated in Table 1. Here, both real and imaginary part of the scalar two-loop $H \to \gamma\gamma$ integral are shown in comparison with the results of Ref. [16]. We consider the domain $q_{thr}^2 < q^2 \leq 100 q_{thr}^2$, where $q_{thr}^2 = 4m_t^2$ ($m_t = 150 GeV$). For q^2 close to the threshold, the integral has a logarithmic singularity, but still we obtain good stability of the approximants, which improves to 8-10 decimals up to $q^2 = 10 q_{thr}^2$ and even for $q^2 = 100 q_{thr}^2$ is still excellent.

For our methods of analytic continuation to work with such high precision, also the Taylor coefficients must be known with high accuracy. In general we should know them analytically and then approximate them with the desired precision. A good example are the coefficients for the $H \to \gamma\gamma$ decay, which can be represented as rational numbers and which for our purpose were approximated with a precision of 45 decimals using REDUCE. In general, however, it turns out that the CPU time needed for indices $n \gtrsim 30$ (in (2) for l = m = 0) is of the order of several hours. Moreover in the arbitrary mass case the length of the expressions even for lower indices becomes enormous and is getting more and more difficult to keep under control. For these reasons it is not possible to obtain analytic expressions for the coefficients with a reasonable effort and that is why we develop a proper algorithm. In the following three sections we formulate this algorithm according to the items specified in Sect.2. FORM will be a guide for the development of the algorithm and a permanent testing tool by comparing results obtained in different manners.

4. The numerator of the integrand.

The first step to be done is the formal Taylor expansion of the integral (4). We choose the following approach: each scalar propagator with an external momentum p is expanded as ($p^2 = 0$)

$$\frac{1}{(k+p)^2 - m_i^2} = \frac{1}{k^2 - m_i^2} \sum_{j=0}^{\infty} \left(\frac{-2kp}{k^2 - m_i^2} \right)^j = \frac{1}{c_i} S_i(k,p) \ .$$

Dealing with the planar and non-planar diagrams (see Fig. 1) simultaneously, as before, we can express the Taylor coefficients under consideration in generalization of (5) – (7) by

$$\begin{aligned} F_n &= D f_{00n} S_1(k_1,p_1) S_2(k_1,p_2) S_3(k_2,p_1) S_4(k_3,p_2) \mid_{p_i=0} \\ &= \sum_{\nu=0}^{n} c_1^{-(n-\nu)} c_3^{-\nu} \sum_{\nu'=0}^{n} c_2^{-(n-\nu')} c_4^{-\nu'} \cdot A^n_{\nu\nu'}(k_1,k_2,k_3) \ . \end{aligned} \tag{19}$$

The differential operator $D f_{00n}$ (3) applied in (19) contains two types of operators

- $\Box_{12}^{n_i}$ with $n_i = n - 2(i-1)$ and
- •• $\Box_{11}^{i-1} \Box_{22}^{i-1}$

such that the sum of the powers $n_i + 2(i-1) = n$. In a first step one has to find a formula for the application of the $\Box_{12}$ operator. The result depends on n_i, the power to which this operator is raised, and the partition of scalar products to which it is applied:

$$tt(n_i, n_i - j_1, n_i - j_2) = \Box_{12}^{n_i} (k_1p_1)^{j_1} (k_2p_1)^{n_i-j_1} (k_1p_2)^{j_2} (k_3p_2)^{n_i-j_2} \ .$$

Performing the differentiation with FORM, we find by inspection

$$tt(l,m,n) \ = \ \sum_{j=0}^{[\frac{m+n}{2}]} t(l,m,n,j) \cdot (k_1^2)^{l-(m+n)+j} (k_1k_2)^{m-j} (k_1k_3)^{n-j} (k_2k_3)^j$$

with

$$t(l,m,n,j) \ = \ \binom{m}{j}\binom{n}{j} j!(l-m)!(l-n)! \frac{l!}{(l-m-n+j)!} \ .$$

Here and in the following we assume inverse powers of factorials of negative arguments to vanish. Counting the powers of p_1 and p_2 in (19), if we wish to calculate $A^n_{\nu\nu'}(k_1,k_2,k_3)$, we need as next

$$2^{(i-1)}(i-1)! dd\left((n-\nu-j_1) \times k_1, (\nu - n_i + j_1) \times k_2\right) = \Box_{11}^{(i-1)} (k_1p_1)^{n-\nu-j_1} (k_2p_1)^{\nu-n_i+j_1} \ , \tag{20}$$

where the function dd of $2(i-1)$ arguments is the totally symmetric tensor, contracted with $n - \nu - j_1$ vectors k_1 and $\nu - n_i + j_1$ vectors k_2:

$$\begin{aligned} dd(m_1,m_2) &= \delta(m_1,m_2) \\ dd(m_1,m_2,m_3,m_4) &= \delta(m_1,m_2)\delta(m_3,m_4) + \delta(m_1,m_3)\delta(m_2,m_4) \\ &\quad + \delta(m_1,m_4)\delta(m_2,m_3) \qquad \text{etc.} \ , \end{aligned}$$

where δ is Kronecker's delta. Similarly for the application of $\Box_{22}$ we write

$$2^{(i-1)}(i-1)!dd\left((n-\nu'-j_2)\times k_1,(\nu'-n_i+j_2)\times k_2\right)=\Box_{22}^{(i-1)}(k_1p_2)^{n-\nu'-j_2}(k_3p_2)^{\nu'-n_i+j_2}\,. \tag{21}$$

The totally symmetric tensor is implemented in FORM 2.2b and can be used for performing the differentiation. For high indices n, however, in this manner the differentiation still requires too much time (for $n=32$ appr. 8 hours on the HP735) and the expressions are too lengthy for practical use. Therefore the idea is to find an analytic expression for (20) and (21) and thus to obtain finally a formula for the coefficients $a_{\nu\nu'}^{n\mu}$ in (7). Indeed $dd(m\times k_1,(l-m)\times k_2)\equiv dd(m,l-m)$ can be written as

$$dd(m,l-m)=\sum_{\substack{j=P(m),\\ \Delta j=2}}^{m} d(l,m,j)(k_1^2)^{\frac{m-j}{2}}(k_2^2)^{\frac{l-m-j}{2}}\,(k_1k_2)^j \tag{22}$$

with $P(m)=(1-(-1)^m)/2$ and

$$d(l,m,j)=\frac{m!(l-m)!}{2^{\frac{l}{2}-j}\left(\frac{l-m-j}{2}\right)!\left(\frac{m-j}{2}\right)!j!}\,.$$

(22) has been verified again by inspection of results from FORM's $dd\_\,(\ldots)$ function (in FORM 2.2b an algorithm was used but not the evaluation of a formula).

Summing over all partitions of scalar products, (19) yields

$$A_{\nu\nu'}^n(k_1,k_2,k_3)$$
$$=(n+1)\;2^{n-1}\frac{\Gamma(d-1)}{\Gamma(n+d-2)\Gamma(n+\frac{d}{2})}\sum_{i=1}^{[\frac{n}{2}]+1}(-1)^{i-1}\frac{(i-1)!}{n_i!}\Gamma(n+\frac{d}{2}-i)f_i$$

with

$$f_i=\sum_{j_1=\max(0,n_i-\nu)}^{\min(n_i,n-\nu)}\;\sum_{j_2=\max(0,n_i-\nu')}^{\min(n_i,n-\nu')}\binom{n-\nu}{j_1}\binom{\nu}{n_i-j_1}\binom{n-\nu'}{j_2}\binom{\nu'}{n_i-j_2}$$
$$\cdot tt(n_i,n_i-j_1,n_i-j_2)\cdot dd(n-\nu-j_1,\nu-n_i+j_1)\cdot dd(n-\nu'-j_2,\nu'-n_i+j_2),$$

where the second dd-factor depends on k_1 and k_3 instead of k_1 and k_2 (see (19)). Rewriting (22) in the following more adequate manner:

$$dd(n-\nu-j_1,\nu-n_i+j_1)=\sum_{\sigma=0}^{[\frac{\nu_j}{2}]}\Theta(\sigma-[\nu_j-(i-1)])$$
$$\cdot\; d\Big(2(i-1),n-\nu-j_1\;,\nu_j-2\sigma\Big)(k_1^2)^{i-1-\nu_j+\sigma}(k_2^2)^{\sigma}(k_1k_2)^{\nu_j-2\sigma}$$

with $\nu_j=\nu-n_i+j_1$ we obtain

$$A_{\nu\nu'}^n(k_1,k_2,k_3)=\sum_{\mu=0}^{\left[\frac{\nu+\nu'}{2}\right]}\sum_{\sigma=0}^{[\frac{\nu}{2}]}\sum_{\tau=0}^{\left[\frac{\nu'}{2}\right]}b_{\nu\nu'}^{n\mu,\sigma\tau}$$
$$\cdot(k_1^2)^{n-(\nu+\nu')+\mu}(k_2^2)^{\sigma}(k_1k_2)^{\nu-\mu-\sigma+\tau}(k_1k_3)^{\nu'-\mu+\sigma-\tau}(k_2k_3)^{\mu-\sigma-\tau}(k_3^2)^{\tau}\;, \tag{23}$$

where the coefficients $b_{\nu\nu'}^{n\mu,\sigma\tau}$ are given by

$$
\begin{aligned}
b^{n\mu,\sigma\tau}_{\nu\nu'} &= (n+1)\, 2^{\lambda-n-1}\, (n-\nu)!\, (n-\nu')!\nu!\, \nu'! \frac{\Gamma(d-1)}{\Gamma(n+d-2)\Gamma(n+\frac{d}{2})} \\
&\frac{2^{2\rho}}{\rho!\sigma!\tau!} \sum_{i=1}^{[\frac{n}{2}]+1} (-4)^{i-1} (i-1)!\, \Gamma(n+\frac{d}{2}-i) \\
&\sum_{j=\max(0,n_i-\nu)}^{\min(n_i,n-\nu)} 2^j \sum_{k=\max(0,n_i-\nu')}^{\min(n_i,n-\nu')} 2^k \frac{1}{(n_i-j-\rho)!(n_i-k-\rho)!(\rho+j+k-n_i)!} \\
&\frac{1}{(\sigma-[j-(n-\nu)+(i-1)])!(\nu_j-2\sigma)!(\tau-[k-(n-\nu')+(i-1)])!(\nu'_k-2\tau)!} ,
\end{aligned}
\tag{24}
$$

$$
\text{with} \qquad \begin{aligned} \lambda &= \nu+\nu'-2\mu\ , \quad \rho=\mu-\sigma-\tau \qquad \text{and} \\ \nu_j &= \nu-n_i+j\ ,\ \nu'_k=\nu'-n_i+k\ . \end{aligned}
$$

First of all it is interesting to note that for the planar diagram (23) reduces with $k_3 = k_2$ to the simpler form (7) if we set

$$
a^{n\mu}_{\nu\nu'} = \sum_{\sigma=0}^{[\frac{\nu}{2}]} \sum_{\tau=0}^{\left[\frac{\nu'}{2}\right]} b^{n\mu,\sigma\tau}_{\nu\nu'}\ , \tag{25}
$$

i.e. the coefficients $a^{n\mu}_{\nu\nu'}$, which in Ref. [1] (appendix A) were only given explicitly as rational numbers for a limited number of indices, are now obtained analytically as five-fold sums. The nice property of the representation (25) is that, once it has been checked against Ref. [1] (which has been done with FORM) for the planar diagram, the coefficients $b^{n\mu,\sigma\tau}_{\nu\nu'}$ for the nonplanar diagram are checked simultaneously.

5. Cancellation of the numerator scalar products.

Since we are here interested in developing an algorithm for the calculation of the Taylor coefficients, the above is the first necessary step. The next is to investigate the possibility of cancelling the numerator scalar products of integration momenta against the bubble propagators c_i $(i = 1, \ldots, 4)$. While in a formula manipulating language this is done blindly e.g. by using the "repeat" command of FORM, here we have to find a detailed prescription if finally our algorithm is to be implemented in terms of a FORTRAN program. Cancellation of scalar products yields "genuine" two-loop bubble integrals which are investigated in terms of recurrence relations in Sect. 6 and factorized one-loop integrals.

At first we study the *planar* diagram. Due to (6), (7) and (11) we can write

$$
\begin{aligned}
F_n = \sum_{\nu,\nu',\mu} a^{n\mu}_{\nu\nu'} \frac{(k_1^2)^{n-(\nu+\nu')+\mu}}{c_1^{n-\nu} c_2^{n-\nu'}} \frac{(k_2^2)^{\mu}}{c_3^{\nu} c_4^{\nu'}} \\
\frac{1}{2^{\nu+\nu'-2\mu}} \left\{ (k_1^2+k_2^2-m_6^2)^{\nu+\nu'-2\mu} - \sum_{\alpha=1,odd}^{\nu+\nu'-2\mu} (k_1^2+k_2^2-m_6^2)^{\nu+\nu'-2\mu-\alpha} (2k_1k_2)^{\alpha-1} \cdot c_6 \right\} .
\end{aligned}
$$

Let us consider the term ($\lambda_\alpha = \nu + \nu' - 2\mu - \alpha; \alpha = 0, \cdots, \nu + \nu' - 2\mu; \lambda_0 \equiv \lambda$)

$$(k_1^2 + k_2^2 - m_6^2)^{\lambda_\alpha} = \lambda_\alpha! \sum_{\beta=0}^{\lambda_\alpha} (-1)^\beta \frac{(m_6)^\beta}{\beta!} \sum_{\gamma=0}^{\lambda_\alpha-\beta} \frac{(k_1^2)^{\lambda-\delta}}{(\lambda-\delta)!} \frac{(k_2^2)^\gamma}{\gamma!},$$

where $\delta = \alpha + \beta + \gamma$ was introduced. Accordingly we have to cancel scalar products in the following combination

$$\frac{(k_1^2)^{n-\mu-\delta}}{c_1^{n-\nu+1} c_2^{n-\nu'+1}} \cdot \frac{(k_2^2)^{\mu+\gamma}}{c_3^{\nu+1} c_4^{\nu'+1} c_5}\ , \tag{26}$$

In both factors of (26) the sum of powers of k_1^2 and k_2^2, respectively, is larger in the denominator than in the numerator so that complete cancellation is possible. Expanding, e.g.,

$$(k_1^2)^{n-\mu-\delta} = \sum_{\kappa=0}^{n-\mu-\delta} \binom{n-\mu-\delta}{\kappa} c_1^{n-\mu-\delta-\kappa} (m_1^2)^\kappa,$$

we have to consider the ratio

$$R_1^\kappa = \frac{c_1^{\nu-\mu-\delta-\kappa-1}}{c_2^{n-\nu'+1}}\ .$$

Depending on the sign of $\nu_1 = \nu - \mu - \delta - \kappa - 1$, the partial fraction decomposition is performed in the following manner (see also Ref. [2], ch. 10):

1) $\nu_1 \geq 0$: in this case we simply have

$$R_1^\kappa = \sum_{i=0}^{\nu_1} \binom{\nu_1}{i} \frac{(m_2^2 - m_1^2)^i}{c_2^{n-\nu'-\nu_1+1+i}}\ ,$$

i.e. there remains no c_1 in the decomposition.

2) $\nu_1 < 0$:

$$\begin{aligned} R_1^\kappa &= \sum_{i=0}^{|\nu_1|-1} (-1)^i \binom{n-\nu'+i}{n-\nu'} \frac{1}{(m_1^2-m_2^2)^{n-\nu'+1+i} c_1^{|\nu_1|-i}} \\ &+ \sum_{i=0}^{n-\nu'} (-1)^{|\nu_1|} \binom{|\nu_1|-1+i}{|\nu_1|-1} \frac{1}{(m_1^2-m_2^2)^{|\nu_1|+i} c_2^{n-\nu'+1-i}}\ . \end{aligned}$$

Similarly we proceed for the k_2^2-dependent part of (26). Expanding

$$(k_2^2)^{\mu+\gamma} = \sum_{\kappa=0}^{\mu+\gamma} \binom{\mu+\gamma}{\kappa} c_4^{\mu+\gamma-\kappa} (m_4^2)^\kappa\ ,$$

we deal with

$$R_2^\kappa = \frac{c_4^{\mu+\gamma-\nu'-\kappa-1}}{c_3^{\nu+1}}\ ,$$

introducing $\nu_2 = \mu + \gamma - \nu' - \kappa - 1$ and performing the partial fraction decomposition as above. Since, however, $c_5 = k_2^2 - m_5^2$ is also k_2-dependent, for each power of $1/c_3$ and $1/c_4$ a further decomposition has to be performed, like e.g.

$$\frac{1}{c_3^{p+1} c_5} = -\sum_{i=0}^{p} \frac{1}{(m_5^2 - m_3^2)^{p+1-i}} \frac{1}{c_3^{i+1}} + \frac{1}{(m_5^2-m_3^2)^{p+1}} \frac{1}{c_5}\ .$$

For $\alpha = 0$ we obtain in this manner "genuine" two-loop integrals of the type (9) which will be dealt with in Sect. 6, while for $\alpha \geq 1$ the factorized one-loop integrals are of the type (see also Ref. [7])

$$\int \frac{d^d k_1 d^d k_2}{(k_1^2 - m_1^2)^{\nu_1}(k_2^2 - m_2^2)^{\nu_2}}(2k_1k_2)^N = \frac{N!}{(\frac{N}{2})!(\frac{d}{2})_{\frac{N}{2}}} I_{\nu_1}^{(N)}(m_1) I_{\nu_2}^{(N)}(m_2)$$

with

$$I_\nu^{(N)}(m) = \int \frac{d^d k}{(k^2 - m^2)^\nu}(k^2)^{\frac{N}{2}} = i^{1-d}\pi^{\frac{d}{2}}(-m^2)^{\frac{d}{2}+\frac{N}{2}-\nu}\frac{\Gamma(\nu - \frac{N}{2} - \frac{d}{2})(\frac{d}{2})_{\frac{N}{2}}}{\Gamma(\nu)} \ .$$

Somewhat differently works the cancellation of the numerator scalar products of the integration momenta for the *non-planar* case since here k_3^2 occurs also in $c_4 = k_3^2 - m_4^2$ which is raised to high inverse powers. Therefore it is advisable to expand the full product of scalar products of different momenta in (23) in terms of squares of integration momenta, i.e. we write with $\lambda_1 = \nu - \mu - \sigma + \tau, \lambda_2 = \nu' - \mu + \sigma - \tau, \lambda_3 = \mu - \sigma - \tau$ and $\lambda = \lambda_1 + \lambda_2 = \nu + \nu' - 2\mu, \Lambda = \lambda_1 + \lambda_2 + \lambda_3 = \nu + \nu' - \mu - \sigma - \tau$

$$\begin{aligned}(k_1k_2)^{\lambda_1}(k_1k_3)^{\lambda_2}(k_2k_3)^{\lambda_3} &= \frac{1}{2^\Lambda}(k_1^2 + k_2^2 - k_3^2)^{\lambda_1}(k_1^2 - k_2^2 + k_3^2)^{\lambda_2}(k_1^2 - k_2^2 - k_3^2)^{\lambda_3} \\ &= \frac{1}{2^\Lambda}\sum_{\alpha=0}^{\Lambda}(k_1^2)^{\Lambda-\alpha} P_\alpha(k_2^2, k_3^2) \ .\end{aligned}$$

A little algebra also allows to expand P_α:

$$P_\alpha(k_2^2, k_3^2) = (-1)^\alpha \sum_{\beta=0}^{\alpha}(-1)^\beta f_{\alpha,\beta}(k_2^2)^\beta(k_3^2)^{\alpha-\beta} \qquad \text{with} \tag{27}$$

$$f_{\alpha,\beta} = \sum_{i=max(0,\alpha-\lambda)}^{min(\alpha,\lambda_3)} \binom{\lambda_3}{i} g_{\alpha,\beta}^i g_{\lambda,\alpha-i}^{\lambda_1} \tag{28}$$

and

$$g_{\alpha,\beta}^k = \sum_{\ell=max(0,k-(\alpha-\beta))}^{min(k,\beta)} (-1)^\ell \binom{\alpha-k}{\beta-\ell}\binom{k}{\ell} \ . \tag{29}$$

k_1^2, e.g., can now be completely cancelled only if

$$n - \mu \geq \Lambda = n - \mu + \theta$$

with $\theta = \nu + \nu' - n - \sigma - \tau \leq 0$. Since obviously we do not always have $\theta \leq 0$ (in contrary to the planar case with $n - (\nu + \nu') + \mu \geq 0$, see (7)) not all k_1^2 can be cancelled, and similarly the same holds for k_2^2 and k_3^2. This means that we will obtain integrals of the type (9) with negative indices for which explicit expressions will be given at the end of Sect. 6. In fact these integrals are also factorized one-loop integrals which only appear in a somewhat different manner than in the planar case.

The following expansions are needed now:

$$\begin{aligned}(k_1^2)^{n-(\nu+\nu')+\mu+\Lambda-\alpha} &= \sum_{\kappa=0}^{n-\sigma-\tau-\alpha}\binom{n-\sigma-\tau-\alpha}{\kappa} c_1^{n-\sigma-\tau-\alpha-\kappa}(m_1^2)^\kappa \\ (k_2^2)^{\sigma+\beta} &= \sum_{\kappa=0}^{\sigma+\beta}\binom{\sigma+\beta}{\kappa} c_3^{\sigma+\beta-\kappa}(m_3^2)^\kappa \quad \text{and} \\ (k_3^2)^{\tau+\alpha-\beta} &= \sum_{\kappa=0}^{\tau+\alpha-\beta}\binom{\tau+\alpha-\beta}{\kappa} c_4^{\tau+\alpha-\beta-\kappa}(m_4^2)^\kappa\end{aligned}$$

and partial fraction decompositions must accordingly be performed for the ratios:

$$\frac{c_1^{\nu-\sigma-\tau-\alpha-\kappa-1}}{c_2^{n-\nu'+1}}\,,\frac{c_3^{\sigma+\beta-\nu-\kappa-1}}{c_5}\quad\text{and}\quad\frac{c_4^{\tau+\alpha-\beta-\nu'-\kappa-1}}{c_6}\,.$$

This works out in analogy to the planar case and will not be discussed in further details.

6. Recurrence relations

As was discussed in Sect.2, the final step of our algorithm consists in the evaluation of the bubble integrals (9) after the Taylor coefficients are finally expressed in terms of these. Their resolution is supposed to be performed in terms of recurrence relations. These were first considered in [17]. One can get such relations from the identity:

$$\int \frac{d^d\ k_1\ d^d\ k_2}{[k_2^2-m_2^2]^\beta}\ \frac{\partial}{\partial k_{1\mu}}\left(\frac{A\ k_{1\mu}-\ B\ k_{2\mu}}{[k_1^2-m_1^2]^\alpha[(k_1-k_2)^2-m_3^2]^\gamma}\right)\equiv 0 \tag{30}$$

with arbitrary constants A and B. Taking, for example, $A=(m_1^2+m_2^2-m_3^2)/2/m_1^2B$ we obtain a recurrence relation for $V_B(\alpha,\beta,\gamma+1)$ and V_B's with their sum of indices equal to $\alpha+\beta+\gamma$. For details on recurrence relations for Feynman diagrams see also Refs. [18], [19]. Explicitly we obtained the following relations for two-loop bubble diagrams with three masses (see [20]):

$$\begin{aligned}
V_B &= \frac{\mathbf{1}^-}{2(j_1-1)m_1^2}\left\{j_3\left[\mathbf{1}^- - \mathbf{2}^- - (m_1^2-m_2^2+m_3^2)\right]\mathbf{3}^+\right. && (31)\\
&+ \left.(2(j_1-1)+j_3-d)\right\}V_B &&\\
V_B &= \frac{\Delta(m_1,m_2,m_3)}{j_2-1}\left\{2j_3m_3^2\ \mathbf{2}^-\left(\mathbf{2}^- - \mathbf{1}^-\right)\mathbf{3}^+\right. && (32)\\
&+ (j_2-1)(m_1^2-m_2^2-m_3^2)\left(\mathbf{3}^- - \mathbf{1}^-\right) &&\\
&+ \left.\left[2(j_2-1)m_3^2+2j_3(m_1^2-m_2^2)+(j_2-1-d)(m_1^2-m_2^2+m_3^2)\right]\mathbf{2}^-\right\}V_B &&\\
V_B &= \frac{\Delta(m_1,m_2,m_3)}{j_3-1}\left\{2j_1m_1^2\ \mathbf{1}^+\left(\mathbf{3}^- - \mathbf{2}^-\right)\mathbf{3}^-\right. && (33)\\
&+ (j_3-1)(m_1^2-m_2^2+m_3^2)\left(\mathbf{2}^- - \mathbf{1}^-\right) &&\\
&+ \left.\left[2(j_3-1)m_1^2+2j_1(m_2^2-m_3^2)+(j_3-1-d)(m_1^2+m_2^2-m_3^2)\right]\mathbf{3}^-\right\}V_B, &&
\end{aligned}$$

where $\mathbf{1}^\pm V_B(j_1,\ldots,m_1,\ldots)\equiv V_B(j_1\pm 1,\ldots,m_1,\ldots)$ etc. and

$$\Delta(m_1,m_2,m_3)=\frac{1}{2m_1^2m_2^2+2m_1^2m_3^2+2m_2^2m_3^2-m_1^4-m_2^4-m_3^4}\quad. \tag{34}$$

The idea of their application is to reduce all integrals to the master integral

$$V_B(1,1,1,m_1,m_2,m_3) \tag{35}$$

and some simple tadpole-integrals, which are obtained when one of the indices is zero. By inspection one observes that applying all three recurrence relations one after the other,

the first index does not increase and the sum of all indices decreases by at least 2 (by 1 in each of the last two steps). In this manner j_2 or j_3 must get zero and the procedure can be stopped.

A particular point we have to observe is that the integrals (9) may diverge, i.e. writing the space-time dimension as $d = 4-2\varepsilon$, we will have poles in ε . Since we intend to develop our algorithm in such a manner that it can be implemented in terms of a FORTRAN program we have to take special care of these poles, i.e. we have to split the integrals (9) into their finite and their divergent part. The master integral (35) and $V_B(1,1,2,m_1,m_2,m_3)$ are the only ones which have a pole of second order (for details see also [1]), and only the integrals $V_B(1,1,n,m_1,m_2,m_3)$ with n > 2 have poles of first order, all others being finite. Thus we write

$$V_B(1,1,n,m_1,m_2,m_3) = F(1,1,n,m_1,m_2,m_3) + \frac{1}{\varepsilon}\, I\ (1,1,n,m_1,m_2,m_3),\ \ n>2. \quad (36)$$

In what follows we will show a path of how to resolve equations (31) - (33). For convenience we drop the masses in the argument list. Using recurrence equation (33) we get

$$\begin{aligned} V_B(1,1,n) \;&= \\ &\frac{\Delta}{n-1}\Big\{2m_1^2\left[V_B(2,1,n-2)-V_B(2,0,n-1)\right]+ \\ &(n-1)(m_1^2-m_2^2+m_3^2)\left[V_B(1,0,n)-V_B(0,1,n)\right]+ \\ &\Big[2(n-1)m_1^2+2(m_2^2-m_3^2)+(n-1-d)(m_1^2+m_2^2-m_3^2)\Big]\,V_B(1,1,n-1)\Big\}\,. \end{aligned}$$

Here $V_B(2,0,n-1), V_B(1,0,n)$ and $V_B(0,1,n)$ are "trivial" and will be substituted explicitly, e.g.

$$V_B(0,m,n) = (-1)^{m+n}\frac{\Gamma\left(m-\frac{d}{2}\right)}{\Gamma\left(m\right)}\frac{i\pi^{n/2}}{m_2^{2\left(m-\frac{d}{2}\right)}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma\left(n\right)}\frac{i\pi^{n/2}}{m_3^{2\left(n-\frac{d}{2}\right)}} \quad (37)$$

with

$$\Gamma(1-\frac{d}{2}) = -\frac{1}{\varepsilon}-1-\varepsilon. \quad (38)$$

This indicates the occurrence of a pole in ε which gives a contribution to I(1,1,n). $V_B(2,1,n-2)$ is found by an application of recurrence equation (31):

$$\begin{aligned} V_B(2,1,n-2) \;&=\; \frac{1}{2m_1^2}\left\{(n-2)\left[V_B(0,1,n-1)-V_B(1,0,n-1)-\right.\right. \\ &\left.\left.(m_1^2-m_2^2+m_3^2)V_B(1,1,n-1)\right]+(n-d)V_B(1,1,n-2)\right\}. \end{aligned}$$

Therefore we can inductively calculate $V_B(1,1,n)$ for all $n \geq 3$ using the master integral $V_B(1,1,1)$. Care has to be taken in evaluating terms of the form $d\ V_B(1,1,n-1)$ on the r.h.s. which yield a finite and an infinite part with $d = 4-2\varepsilon$. Using recurrence equation (32), we get moreover

$$\begin{aligned} &V_B(1,m,n) = \\ &\frac{\Delta}{m-1}\Big\{2nm_3^2\left[V_B(1,m-2,n+1)-V_B(0,m-1,n+1)\right]+ \\ &(m-1)(m_1^2-m_2^2-m_3^2)\left[V_B(1,m,n-1)-V_B(0,m,n)\right]+ \\ &\Big[2(m-1)m_3^2+2n(m_1^2-m_2^2)+(m-1-d)(m_1^2-m_2^2+m_3^2)\Big]\,V_B(1,m-1,n)\Big\} \end{aligned}$$

so that $V_B(1,m,n)$ is computable, and finally an application of recurrence equation (31) leads to

$$
\begin{aligned}
&V_B(l,m,n) = \\
&\frac{1}{2m_1^2(l-1)}\left\{n\left[V_B(l-2,m,n+1) - V_B(l-1,m-1,n+1) - \right.\right. \\
&\left.\left.(m_1^2 - m_2^2 + m_3^2)V_B(l-1,m,n+1)\right] + (2(l-1)+n-d)V_B(l-1,m,n)\right\} \ .
\end{aligned}
$$

Implementing the recurrence relations numerically, it is advisable, in order to take properly into account the poles in ε, to produce explicit relations for the lower indices by means of FORM. For the $V_B(1,1,n)$, e.g., the $\frac{1}{\varepsilon^2}$ pole from the master integral causes "extra" divergences and for the $V_B(l,m,n)$ divergences come in for lower indices whenever a $V_B(1,1,n)$ is encountered in a recurrence relation while for higher indices these integrals are finite as can also directly be seen from (9).

Apart from that, the recurrence relations are implemented in exactly the order as described above: first $V_B(1,1,n)$ for all n, secondly $V_B(1,m,n)$ and finally $V_B(l,m,n)$. In each of these cases the recurrence has been implemented for all $l+m+n \leq J$ ($l,m,n \geq 1$; for negative indices see below), where the large integer J is to be chosen according to the number of Taylor coefficients needed.

Of course one may have doubts about the numerical stability of such a recursive approach, in particular if one knows that indeed the Taylor coefficients must be calculated with high precision as mentioned in Sect. 2. Ordinary FORTRAN double precision will clearly not do the job. We used the multiple precision package written by D.H.Bayley [21], which also provides an automatic translator for any FORTRAN program. The requested precision is here defined at the beginning of the program. With this package we used for $J = 62$ and 100 decimals precision 42 seconds on a HP735 (49 seconds for 150 decimals precision). The numerical results were tested against numerics performed with REDUCE for explicit expressions of the integrals in the equal mass case. In principle, however, the best precision test will always be to increase the number of decimals used in the calculation. In this manner we can be sure to have an effective algorithm for the calculation of two-loop integrals.

To conclude this sections we give an explicit formula for $V_B(\alpha,\beta,\gamma)$ for the case that one of the indices is negative. In this case the V_B's can be reduced to factorized one-loop integrals, which makes the application of the above recurrence relations superfluous.

$$
\begin{aligned}
&V_B(\alpha,\beta,\gamma) = (-1)^{(\alpha+\beta+\gamma)} \int \frac{d^dk_1 \ d^dk_2 \ [(k_1-k_2)^2 - m_3^2]^{|\gamma|}}{(k_1^2-m_1^2)^\alpha (k_2^2-m_2^2)^\beta} = \qquad (39) \\
&(i\pi^{\frac{d}{2}})^2 |\gamma|! \sum_{l=0}^{[|\gamma|/2]} \frac{(\frac{d}{2})_l}{l!} \sum_{r=max(0,2l+\alpha-|\gamma|-1)}^{\alpha-1} \sum_{q=0}^{\beta-1} \frac{\Gamma(1+r-l-\frac{d}{2})\Gamma(1+q-l-\frac{d}{2})}{r!q!(\alpha-1-r)!(\beta-1-q)!} \\
&\qquad \frac{s^{|\gamma|-2l-\alpha-\beta+r+q+2}(m_1^2)^{(\frac{d}{2}+l-r-1)}(m_2^2)^{(\frac{d}{2}+l-q-1)}}{(|\gamma|-2l-\alpha-\beta+r+q+2)!} \ ,
\end{aligned}
$$

where $\gamma < 0$ and $s = -m_1^2 - m_2^2 + m_3^2$.

7. Conclusions

An effective method to calculate Feynman diagrams has been developed in [1]. In the present work we have been able to work out details of an algorithm, which will finally allow to elaborate the above method into a "package" for the evaluation of two-loop three-point functions and possibly beyond. The essential new points worked out here are described in Sects. 4 and 6: deriving an explicit formula for the numerators in the integral representation of the Feynman diagrams's Taylor coefficients and demonstrating the possibility to evaluate the recursion relations for the bubble integrals numerically by means of the multiple precision package of [21]. In the development of this algorithm FORM [2] has been the main tool in the formulae manipulation.

Acknowledgments

We are very much indebted to J. Vermaseren to indicate to us on this conference the new possibility of FORM, namely the evaluation of the totally symmetric tensor as special function, which enabled us to develop our algorithm in the presented form. We are also grateful to W. Koepf for discussions on the analytic resolution of the recurrence relations by means of non-commutative Gröbner basis techniques. Thanks as well to L. Avdeev for helpfull discussions.

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